

Hamiltonian LGT in the complete Fourier analysis basis.

G. Burgio^a, R. De Pietri^b, H. A. Morales-Técotl^c, L. F. Urrutia^d and J. D. Vergara^d.

^aDipartimento di Fisica, Università di Parma and INFN, Gruppo Collegato di Parma, Parma, Italy

^bCentre de Physique Théorique CNRS, Case 907 Campus de Luminy, F-13288 Marseille Cedex 9, France

^cDepartamento de Física, Universidad Autónoma Metropolitana Iztapalapa, A. Postal 55-534, 09340 México, D.F.

^dDepartamento de Física de Altas Energías, Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A. Postal 70-543, 04510 México D.F.

The main problem in the Hamiltonian formulation of Lattice Gauge Theories is the determination of an appropriate basis avoiding the over-completeness arising from Mandelstam relations. We short-cut this problem using Harmonic analysis on Lie-Groups and intertwining operators formalism to explicitly construct a basis of the Hilbert space. Our analysis is based only on properties of the tensor category of Lie-Group representations. The Hamiltonian of such theories is calculated yielding a sparse matrix whose spectrum and eigenstates could be exactly derived as functions of the coupling g^2 .

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1. HAMILTONIAN LATTICE GAUGE THEORY

The Hamiltonian formalism for lattice gauge theories (LGT) in $(d+1)$ dimensions [1] is constructed associating gauge field variables $U_k(\mathbf{x}) \in G$ to each link $(\mathbf{x}, \mathbf{x} + a\mathbf{e}_k)$ of a hypercubic periodic lattice of period aL . The corresponding Hilbert space \mathcal{H} is defined by the gauge invariant square integrable functions $\psi(U) = \psi(U^\gamma) = \psi(\{U_k^\gamma(\mathbf{x})\})$ on the tensor product of $d \cdot L^d$ copies of the gauge group G . Gauge transformations act as $U_k(\mathbf{x}) \longrightarrow U_k^\gamma(\mathbf{x}) = \gamma^{-1}(\mathbf{x} + a\mathbf{e}_k)U_k(\mathbf{x})\gamma(\mathbf{x})$. The variables conjugated to $U_k(\mathbf{x})$ are the outgoing/ingoing electric fields $E_{\pm k}^\alpha(\mathbf{x})$ from the lattice point \mathbf{x} in the directions $\mathbf{e}_{\pm k}$.

The standard Hamiltonian operator is

$$\hat{H} = \frac{g^2}{2a^{d-2}} \sum_{\mathbf{x}, k} q_{\alpha\beta} E_k^\alpha(\mathbf{x}) E_k^\beta(\mathbf{x}) + \sum_P V(U_P), \quad (1)$$

where $q_{\alpha\beta}$ is the Cartan metric, the sum over P ranges over all unoriented plaquettes, U_P is the plaquette variable and

$$V(U_P) = \frac{a^{d-4}}{g^2} \left[1 - \frac{U_P + U_P^*}{2\dim(U)} \right]. \quad (2)$$

Such choice is not unique since the only condition on the magnetic term potential is $V(U_P) \simeq \frac{a^4}{2} \text{Tr}[F_P^2]$.

2. THE SPIN NETWORK BASIS

A classical result of representation theory gives a nice way of constructing a basis of LGT Hilbert space. In fact, the set $\mathcal{RG} = \{\mathcal{R}^j \mid j \in J[G]\}$ of all the unitary inequivalent representations of a compact group G is numerable and all the representations \mathcal{R}^j are finite dimensional. Choosing an orthonormal basis for each representation \mathcal{R}^j , the matrix elements $D^{(j)\alpha}_\beta(U)$ ($\alpha, \beta = 1, \dots, \dim(\mathcal{H}^j)$) of all the representations \mathcal{R}^j are a numerable orthonormal basis of $\mathcal{L}^2[G, dU]$. This result, known as the Peter-Weyl theorem [5], implies that each vector of \mathcal{H} can be written as

$$\psi(U) = \prod_{\mathbf{x}} \prod_{k=1}^d \sum_{j_{\mathbf{x}}^k \in J[G]} \sum_{\alpha_{\mathbf{x}}^k, \beta_{\mathbf{x}}^k=1}^{\dim(j_{\mathbf{x}}^k)} \left[D^{(j_{\mathbf{x}}^k)\alpha_{\mathbf{x}}^k}_{\beta_{\mathbf{x}}^k}(U) \times c^{(j_{(1)} \dots j_{(N_{1k}))\beta_{(1)} \dots \beta_{(N_{1k})}}_{\alpha_{(1)} \dots \alpha_{(N_{1k})}} \right], \quad (3)$$

where only gauge invariant combinations should be taken into account.

The implementation of gauge invariance turns into a set of constraints on the coefficients c . In particular the c 's should factorize in products of group invariant tensors associated to the different lattice sites \mathbf{x} .

The concept of invariant tensor is better expressed by the notion of intertwining operators. By definition, an operator l connecting the Hilbert space of two representations, \mathcal{R} and \mathcal{R}' , is an intertwining operator if $\mathsf{l} \cdot T(U) = T'(U) \cdot \mathsf{l}$, for every U in G . The set of all intertwining operators $\mathcal{I}(\mathcal{R}, \mathcal{R}')$ is a vector subspace of all the linear operators connecting the Hilbert space of the two representations \mathcal{R} and \mathcal{R}' . This concept gives the coordinate free definition of the generalized Clebsh-Gordan coefficients of Yutsis-Levinson-Vanagas [3] which are the matrix elements of these operators on the chosen basis. The integral of the product of K representations decomposes according to

$$\int dU \prod_{k=1}^K D_{\beta_k}^{(j_k)\alpha_k}(U) = \sum_{\pi} \frac{|\mathbb{I}^{(j_1 \dots j_K)}|_{\beta_1 \dots \beta_K}^{\alpha_1 \dots \alpha_K}}{|\mathbb{I}^{(j_1 \dots j_K)}|_{\gamma_1 \dots \gamma_K}^{\gamma_1 \dots \gamma_K}}, \quad (4)$$

where $|\mathbb{I}^{(j_1 \dots j_K)}| \in \mathcal{I}(\mathcal{R}^{j_1} \otimes \dots \otimes \mathcal{R}^{j_K}, \emptyset)$, $|\mathbb{I}^{(j_1 \dots j_K)}|$ is its adjoint and the sum is extended over a complete orthogonal basis of $\mathcal{I}(\mathcal{R}^{j_1} \otimes \dots \otimes \mathcal{R}^{j_K}, \emptyset)$.

Summarizing, Peter-Weyl theorem and gauge invariance leads to the *spin network* basis elements

$$\psi_{\vec{j}, \vec{\pi}}(U) = \prod_{\mathbf{x}} \prod_{k=1}^d \sum_{\alpha_{\mathbf{x}}^k, \beta_{\mathbf{x}}^k=1}^{\dim(j_{\mathbf{x}}^k)} \left[D_{\beta_{\mathbf{x}}^k}^{(j_{\mathbf{x}}^k)\alpha_{\mathbf{x}}^k}(U_k(\mathbf{x})) \cdot |\mathbb{I}_{\mathbf{x}}^{[\pi_{\mathbf{x}}]}|_{\alpha_{\mathbf{x}-d}^1, \dots, \alpha_{\mathbf{x}-d}^d}^{\beta_{\mathbf{x}}^1, \dots, \beta_{\mathbf{x}}^d} \right]. \quad (5)$$

3. MATRIX ELEMENTS OF THE HAMILTONIAN OPERATOR

Computing the action of the Hamiltonian operator (1) on the *spin-networks basis* simply reduces to tracing intertwining operators. In fact, the basis vectors (5) are eigenstates of the kinetic term, while the potential (magnetic) term is realized as

a multiplicative operator. Explicitly

$$\begin{aligned} \langle \vec{j}', \vec{\pi}' | \hat{H} | \vec{j}, \vec{\pi} \rangle &= \frac{-a^{d-4}}{2 g^2 \dim(U)} \sum_{\mathbf{y}} \sum_{r < s=1 \dots d} \times \\ &\times \left(\langle \vec{j}', \vec{\pi}' | U_{\mathbf{y}, r, s} | \vec{j}, \vec{\pi} \rangle + \langle \vec{j}, \vec{\pi} | U_{\mathbf{y}, r, s} | \vec{j}', \vec{\pi}' \rangle \right) \\ &+ \left(\frac{g^2}{2 a^{d-2}} \sum_{\mathbf{x}} \sum_{k=1}^d C_2[j_{\mathbf{x}}^2] + \frac{a^{d-4}}{g^2} N_P \right) \delta_{\vec{j}}^{\vec{j}'} \delta_{\vec{\pi}}^{\vec{\pi}'}, \end{aligned}$$

where the only non diagonal terms are given by the expectation values of the plaquette operator. These are given in equation (26) of Ref.[4] as traces of intertwining operators. In this way the computation amounts to the evaluation of specific Wigner's nJ -symbols, that further reduce to $6J$ -symbols only.

An important property of the Hamiltonian matrix in the *spin-network* basis is that it is sparse.

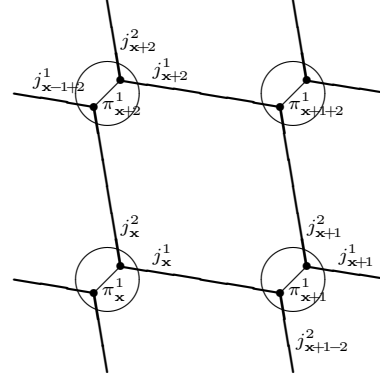


Figure 1. A spin network in the case of a 2 dimensional lattice is parametrized by an irreducible representation associated to each link $j_{\mathbf{x}}^{1,2}$ and an irreducible representation $\pi_{\mathbf{x}}^1$ parametrizing the irreducible tensor associated to each lattice site.

For example, in a two dimensional lattice and gauge group $SU(2)$ the spin-network basis elements are characterized by three half integer associated to each lattice site (see Fig. 1). The matrix elements of the plaquette $U_{\mathbf{y}, 1, 2}$ are different from zero only if all the six primed and

un-primed $j_{\mathbf{x}}^1, j_{\mathbf{x}}^2, \pi_{\mathbf{x}+1}^1, j_{\mathbf{x}+1}^2, \pi_{\mathbf{x}+2}^1, j_{\mathbf{x}+2}^2$, differ by a half integer for $\mathbf{x}=\mathbf{y}$, being zero otherwise. The explicit expression of the matrix elements of the plaquette operator are:

$$\begin{aligned} \langle \vec{j}', \vec{\pi}' | \mathbf{U}_{\mathbf{y},1,2} | \vec{j}, \vec{\pi} \rangle = \\ = \frac{(-1)^{\sum_{i=1}^n \left(|\epsilon_i - \epsilon_{i+1}| + \frac{C_{\mathbf{y}}^i}{2} \right)}}{\sqrt{\prod_{i=1}^n (2X_{\mathbf{y}}^i + 1) (2Y_{\mathbf{y}}^i + 1)}} \times \\ \times \prod_{i=1}^n R \left[\begin{array}{cc} X_{\mathbf{y}}^i & X_{\mathbf{y}}^{i+1} \\ Y_{\mathbf{y}}^i & Y_{\mathbf{y}}^{i+1} \end{array}, C_{\mathbf{y}}^i \right] \end{aligned} \quad (6)$$

where $\epsilon_i = X_{\mathbf{y}}^i - Y_{\mathbf{y}}^i = \pm \frac{1}{2}$,

$$\begin{array}{lll} X_{\mathbf{y}}^1 = j_{\mathbf{x}}^1, & Y_{\mathbf{y}}^1 = j_{\mathbf{x}}^{1'}, & C_{\mathbf{y}}^1 = \pi_{\mathbf{x}}^1, \\ X_{\mathbf{y}}^2 = j_{\mathbf{x}}^2, & Y_{\mathbf{y}}^2 = j_{\mathbf{x}}^{2'}, & C_{\mathbf{y}}^2 = j_{\mathbf{x}-1+2}^1, \\ X_{\mathbf{y}}^3 = \pi_{\mathbf{x}+2}^1, & Y_{\mathbf{y}}^3 = \pi_{\mathbf{x}+2}^{1'}, & C_{\mathbf{y}}^3 = j_{\mathbf{x}+2}^2, \\ X_{\mathbf{y}}^4 = j_{\mathbf{x}+2}^1, & Y_{\mathbf{y}}^4 = j_{\mathbf{x}+2}^{1'}, & C_{\mathbf{y}}^4 = \pi_{\mathbf{x}+1+2}^1, \\ X_{\mathbf{y}}^5 = j_{\mathbf{x}+1}^2, & Y_{\mathbf{y}}^5 = j_{\mathbf{x}+1}^{2'}, & C_{\mathbf{y}}^5 = j_{\mathbf{x}+1}^1, \\ X_{\mathbf{y}}^6 = \pi_{\mathbf{x}+1}^1, & Y_{\mathbf{y}}^6 = \pi_{\mathbf{x}+1}^{1'}, & C_{\mathbf{y}}^6 = j_{\mathbf{x}+1-2}^2 \end{array}$$

and $R \left[\begin{array}{cc} X_{\mathbf{y}}^i & X_{\mathbf{y}}^{i+1} \\ Y_{\mathbf{y}}^i & Y_{\mathbf{y}}^{i+1} \end{array}, C_{\mathbf{y}}^i \right]$ is equal to

$$\sqrt{\frac{1-2C_{\mathbf{y}}^i+X_{\mathbf{y}}^i+X_{\mathbf{y}}^{i+1}+Y_{\mathbf{y}}^i+Y_{\mathbf{y}}^{i+1}}{2} \frac{3+2C_{\mathbf{y}}^i+X_{\mathbf{y}}^i+X_{\mathbf{y}}^{i+1}+Y_{\mathbf{y}}^i+Y_{\mathbf{y}}^{i+1}}{2}}$$

for $|\epsilon_i - \epsilon_{i+1}| = 0$ and

$$\sqrt{\frac{1+2C_{\mathbf{y}}^i+X_{\mathbf{y}}^i-X_{\mathbf{y}}^{i+1}+Y_{\mathbf{y}}^i-Y_{\mathbf{y}}^{i+1}}{2} \frac{1+2C_{\mathbf{y}}^i-X_{\mathbf{y}}^i+X_{\mathbf{y}}^{i+1}-Y_{\mathbf{y}}^i+Y_{\mathbf{y}}^{i+1}}{2}}$$

for $|\epsilon_i - \epsilon_{i+1}| = 1$

4. CONCLUSIONS

In this work we have shown how *harmonic analysis on compact Lie-Groups* together with *intertwining operators formalism* provide a useful basis of the Hilbert space of LGT and an explicit expression for matrix elements of the Hamiltonian operator. Moreover, the latter can be expressed in terms of Wigner's nJ -symbols. Such elements are well known for $SU(2)$ (see for example [3]). Explicit values for the nJ -symbol for unitary groups are found in [6].

Our results are derived in terms of the knowledge of : 1) the full set of irreducible representations of the group, 2) a basis on the space of intertwining operators. The generalization to gauge

theories coupled to matter is therefore straightforward.

The construction does not depend on the gauge group but only on its tensor category properties.

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